Variance

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

for discrete variables

$$\sigma^2 = \langle i^2 \rangle - \langle i \rangle^2$$

$$\sigma^2 = \sum_{i=1}^{t} i^2 p(i) - \left[\sum_{i=1}^{t} i\, p(i) \right]^2$$

for continuous variables

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$\sigma^2 = \int_a^b x^2 p(x)\,dx - \left[\int_a^b x\, p(x)\,dx \right]^2$$

the standard deviation is given by:

$$S_d = \sqrt{\sigma^2} \equiv \sigma$$

Stirling's Approximation

$$N! \cong \left(\frac{N}{e}\right)^N$$

{ get this from a Taylor-McLaurin series }

$$\downarrow$$

$$\ln(N!) \cong \ln\left[\frac{N}{e}\right]^N = \ln N^N - \ln e^N$$

$$\downarrow$$

$$\ln(N!) \cong N\ln N - N\ln e$$

$$\downarrow$$

$$\ln(N!) \cong N\ln N - N = N(\ln N - 1)$$

$$\downarrow \text{ for very large } N \quad \ln N \gg 1$$

$$\ln(N!) \cong N\ln N$$

Binomial Theorem

$$(x+y)^N = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!}\, y^{N-n_1}\, x^{n_1}$$

$\Big\downarrow$ Let $y = 1$

$$(1+x)^N = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!}\, x^{n_1}$$

we make use of the above equation to determine $\langle n_1 \rangle$ and $\langle n_1^2 \rangle$ for a binomial dist'n

Let n_1 = # successes of a coin-flip $\left(p = \frac{1}{2}\right)$ with N trials

$$P_{n_1} = p^{n_1}\, p^{(N-n_1)} \frac{N!}{n_1!\,(N-n_1)!} = \left(\frac{1}{2}\right)^N \frac{N!}{n_1!\,(N-n_1)!}$$

note: $$\sum_{n_1=0}^{N} P_{n_1} = \sum_{n_1=0}^{N} \frac{1}{2^N} \frac{N!}{n_1!\,(N-n_1)!} = 1$$

$$\Downarrow$$

$$\sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!} = 2^N$$

just as if we set $x = 1$ in the equation above

Now, let's find $\langle n_1 \rangle = \sum_{n_1=0}^{N} n_1 P_{n_1}$

$$\langle n_1 \rangle = \sum_{n_1=0}^{N} n_1 p^{n_1} (1-p)^{N-n_1} \frac{N!}{n_1!\,(N-n_1)!}$$

$$\Big\downarrow p = \tfrac{1}{2}$$

$$\langle n_1 \rangle = \sum_{n_1=0}^{N} n_1 \left(\frac{1}{2}\right)^{N} \frac{N!}{n_1!\,(N-n_1)!}$$

$$= \frac{\displaystyle\sum_{n_1=0}^{N} n_1 \frac{N!}{n_1!\,(N-n_1)!}}{2^N}$$

From the binomial theorem

$$(1+x)^N = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!} x^{n_1}$$

$$\frac{d(1+x)^N}{dx} = \frac{d}{dx}\left[\sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!} x^{n_1}\right]$$

$$N(1+x)^{N-1} = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!} n_1 x^{n_1-1}$$

$$\Big\downarrow \text{let } x = 1$$

$$N 2^{N-1} = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!} n_1$$

$$\therefore \quad \langle n_1 \rangle = \frac{\sum_{n_1=0}^{N} n_1 \frac{N!}{n_1!(N-n_1)!}}{2^N}$$

$$= \frac{N\,2^{N-1}}{2^N}$$

$$\downarrow$$

$$\langle n_1 \rangle = \frac{N}{2}$$

Find $\langle n_1^2 \rangle$ similarly $\Rightarrow$ $\langle n_1^2 \rangle = \frac{\sum_{n_1=0}^{N} n_1^2 \frac{N!}{n_1!(N-n_1)!}}{2^N}$

$$\frac{d^2}{dx^2}(1+x)^N = \frac{d}{dx}\left(N(1+x)^{N-1}\right) = N(N-1)(1+x)^{N-2}$$

$$= \frac{d^2}{dx^2}\left[\sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} x^{n_1}\right]$$

$$= \sum_{n_1=0}^{N} (n_1)(n_1-1)\frac{N!}{n_1!(N-n_1)!} x^{n_1}$$

$$\Downarrow \; x = 1$$

$$N(N-1)\,2^{N-2} = \sum_{n_1=0}^{N} n_1^2 \frac{N!}{n_1!(N-n_1)!} - \sum_{n_1=0}^{N} n_1 \frac{N!}{n_1!(N-n_1)!}$$

$$\downarrow$$

$$N(N-1)\,2^{N-2} = 2^N \langle n_1^2 \rangle - 2^N \langle n_1 \rangle$$

$$\frac{N(N-1)}{4} = \langle n_1^2 \rangle - \langle n_1 \rangle$$

$$\downarrow$$

$$\frac{N(N-1)}{4} + \frac{N}{2} = \langle n_1^2 \rangle$$

$$\langle n_1^2 \rangle = \frac{N^2}{4} - \frac{N}{4} + \frac{2N}{4} = \frac{N^2}{4} + \frac{N}{4}$$

$$\langle n_1^2 \rangle - \langle n_1 \rangle^2 = \frac{N^2}{4} + \frac{N}{4} - \frac{N^2}{4} = \frac{N}{4}$$

$$\Downarrow$$

$$\sigma^2 = \frac{N}{4}$$

$$\sigma = \frac{\sqrt{N}}{2}$$

ratio $\cancel{\frac{\sigma}{\langle n_1 \rangle} = \frac{1}{\sqrt{N}} \to 0 \text{ as } N \to \infty}$

$$\lim_{N \to \infty} \frac{\sigma}{\langle n_1 \rangle} = \lim_{N \to \infty} \frac{\frac{\sqrt{N}}{2}}{\frac{N}{2}} = \lim_{N \to \infty} \frac{1}{\sqrt{N}} \to \phi$$

as N gets large only the mean (expected) value is observed

For binomial distribution

Let n_1 = # successes, $p = \frac{1}{2}$, N trials

$$P_{n_1} = \frac{\frac{N!}{n_1!\,(N-n_1)!}}{2^N} = \frac{W(n_1, N)}{2^N}$$

from a well-known theorem

$$(x+y)^N = \sum_{n_1=0}^{N} \frac{N!\, x^{(N-n_1)}\, y^{n_1}}{n_1!\,(N-n_1!)}$$

N is an integer greater than $\emptyset$

$$\Big\downarrow \text{ let } x = 1$$

$$(1+y)^N = \sum_{n_1=0}^{N} \frac{N!}{n_1!\,(N-n_1)!}\, y^{n_1}$$

$$\Big\downarrow$$

$$(1+y)^N = \sum_{n_1=0}^{N} W(n_1, N)\, y^{n_1}$$

$$\Big\downarrow \text{ Let } y = 1$$

$$2^N = \sum_{n_1=0}^{N} W(n_1, N)$$

$$\therefore \quad P_{n_1} = \frac{W(n_1, N)}{\sum_{n_1=0}^{N} W(n_1, N)}$$

we now determine $\langle n_1 \rangle$

$$\langle n_1 \rangle = \sum_{n_1=0}^{N} n_1 P_{n_1} = \frac{\sum_{n_1=0}^{N} n_1 W(n_1, N)}{\sum_{n_1=0}^{N} W(n_1, N)}$$

examine more closely the term

$$n_1 W(n_1, N) = \frac{n_1 N!}{n_1! (N-n_1)!} = \frac{N!}{(n_1-1)! (N-n_1)!}$$

with a little manipulation we can make the right-hand side look like a binomial multiplicity.

$$\frac{N!}{(n_1-1)! (N-n_1)!} = \frac{N (N-1)!}{(n_1-1)! \left[(N-1)-(n_1-1)\right]!} = N\, W\big((n_1-1), (N-1)\big)$$

the numerator in our expression for $\langle n_1 \rangle$ besomes

$$\sum_{i=1}^{N} N\, W\big((n_1-1), (N-1)\big) = N \sum_{i=1}^{N} W\big((n_1-1), (N-1)\big)$$

$$= N\, 2^{N-1}$$

$$\therefore \quad \langle n_1 \rangle = \frac{N\,2^{N-1}}{2^N} = \frac{N}{2}$$

this means that for N flips of an unbiased coin the average # heads (or # tails) $\equiv n_1$ is exactly half (i.e. $\frac{N}{2}$)

You will show in the HW that if you optimize $W(n_1, N)$ with respect to n_1 (same as optimizing $\ln W(n_1, N)$), the value n_1^* that maximizes $\ln W(n_1, W)$ is:

$$n_1^* = \langle n_1 \rangle$$

Highest Probability Outcomes $\Rightarrow$ Highest Multiplicity

Ex: 4 coin flips

n_H	$W(n_H, 4)$	$p(n_H)$
0	1	$\frac{1}{16} = 0.0625$
1	4	$\frac{1}{4} = 0.2500$
2	6	$\frac{3}{8} = 0.3750$
3	4	$\frac{1}{4} = 0.2500$
4	1	$\frac{1}{16} = 0.0625$

$W_{total} = 16 = 2^4$

10 coin flips

	n_H	$W(n_H, 10)$	$p(n_H)$
(10)	0	1	0.0009765625
(9)	1	10	0.009765625
(8)	2	45	0.0439453125
(7)	3	120	0.1171875
(6)	4	210	0.205078125
	5	252	0.24609375

Simple models for physical systems

- gases exert pressure on container walls $\Rightarrow$ gases expand to fill volume
- model (in 1-D) for this system are 3 balls (gas particles) in boxes (total boxes = volume)
    - constraint on this system (model) is that there can only be 1 ball per box
- smallest volume = 3 boxes and expand from here
    - if we assume that the probability of filling any one box is the same as any other then we have Bernoulli trials with success = the ball in a box
    - N = # boxes $\Rightarrow$ volume

      3 = # successes = filled boxes

      $N-3$ = # empty boxes = failures
- highest probability $\Rightarrow$ highest multiplicity as we have seen with our coin flip example (also Bernoulli trials)

N	W	# configurations w/3 ball clusters
3	1	(1)
4	4	(2)
5	10	(3)
6	20	(4)
7	35	(5)
8	56	(6)
9	84	(7)
10	120	(8)

$$W = \frac{N!}{3!\,(N-3)!}$$

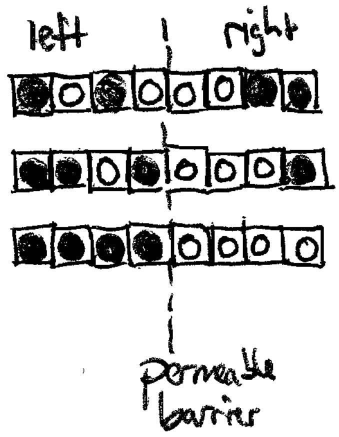
as N ↑ so does W, and the probability of having 3-ball clusters goes down

Maximum multiplicity ⇒ balls spread out over all boxes

boxes ⇒ volume

- materials diffuse and mix
- in this model we fix the volume (# boxes) and we have 2 colors of balls ● and ○
- we split the number of boxes with a "permeable" barrier that allows balls to move from one side of barrier to the other
- like previous model only 1 ball per box and probability of a ● filling a box is the same as a ○ filling a box (all boxes are filled)

boxes = 8   # ● = 4   # ○ = 4



Case A

Case B

Case C

- for this problem we partition the 8 boxes into 2 sets of 4 boxes left side and right side

- we now have 2 Bernoulli trial sets
- let a box with ● = success

$\therefore$ for case A

$$W(\text{left}) = \frac{4!}{2!\,2!} \qquad W(\text{right}) = \frac{4!}{2!\,2!}$$

from the multiplication rule for probabilities (independent trials)

$$p(\text{left}\cdot\text{right}) = p(\text{left})\, p(\text{right})$$

$$\Downarrow$$

$$W(\text{left}\cdot\text{right}) = W(\text{left})\, W(\text{right}) = W$$

Case	W
A	$\frac{4!}{2!\,2!}\cdot\frac{4!}{2!\,2!} = 36$
B	$\frac{4!}{3!\,1!}\cdot\frac{4!}{1!\,3!} = 16$
C	1